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2002 J. Phys. A: Math. Gen. 35 377

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Geometric phases for neutral and charged particles in a time-dependent magnetic field

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Received 23 July 2001, in final form 24 October 2001

Published 4 January 2002

Online at stacks.iop.org/JPhysA/35/377

Abstract

It is well known that any cyclic solution of a spin $1/2$ neutral particle moving in an arbitrary magnetic field has a nonadiabatic geometric phase proportional to the solid angle subtended by the trace of the spin. For neutral particles with higher spin, this is true for cyclic solutions with special initial conditions. For more general cyclic solutions, however, this does not hold. As an example, we consider the most general solutions of such particles moving in a rotating magnetic field. If the parameters of the system are appropriately chosen, all solutions are cyclic. The nonadiabatic geometric phase and the solid angle are both calculated explicitly. It turns out that the nonadiabatic geometric phase contains an extra term in addition to the one proportional to the solid angle. The extra term vanishes automatically for spin $1/2$. For higher spin, however, it depends on the initial condition. We also consider the valence electron of an alkaline atom. For cyclic solutions with special initial conditions in an arbitrary strong magnetic field, we prove that the nonadiabatic geometric phase is a linear combination of the two solid angles subtended by the traces of the orbit and spin angular momenta. For more general cyclic solutions in a strong rotating magnetic field, the nonadiabatic geometric phase also contains extra terms in addition to the linear combination.

PACS numbers: 03.65.Ta, 03.65.Vf

1. Introduction

The motion of spin (especially spin $1/2$) in a rotating magnetic field is a rather classical problem in quantum mechanics, which was discussed in the textbook [1]. Nevertheless, the

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problem has received much attention in recent years [2–7]. The reason may be that the Schrödinger equation for the problem can be solved analytically, and thus it serves as a good example for manifesting the notions of adiabatic geometric phase, nonadiabatic geometric phase for cyclic and noncyclic motions [8–13]. Moreover, it is relevant to some problems in condensed matter physics [7].

Cyclic solutions with special initial conditions were widely discussed in the above cited papers, for both spin $1/2$ and higher ones. It is well known that the nonadiabatic geometric phase for such solutions is always proportional to the solid angle subtended by the trace of the spin (more exactly, the mean value of the spin). Because the nonadiabatic geometric phase is a geometric object, and because the result holds for any cyclic solution of spin $1/2$ in an arbitrarily varying magnetic field [13, 14], one may become confident that it is also true for higher spin. It is indeed true for special cyclic solutions in a rotating magnetic field as just mentioned. For more general cyclic solutions and more general magnetic fields, however, the result was neither proved nor refuted. In fact, the nonadiabatic geometric phase for solutions with more general initial conditions was calculated by some authors only for spin $1/2$ [7, 14]; that for higher spin was, however, not studied to our knowledge.

In this paper we consider both neutral and charged particles. In section 2 we consider neutral particles with general spin and with magnetic moment moving in a rotating magnetic field. The Schrödinger equation for the problem can be solved exactly by making use of a time-dependent unitary transformation. Solutions with special initial conditions are cyclic and have been studied in detail [2]. When the parameters of the system are appropriately chosen, all solutions are cyclic. These solutions were not discussed in detail previously. We calculate the nonadiabatic geometric phase for such solutions. The solid angle subtended by the trace of the spin is also calculated explicitly. It turns out that the nonadiabatic geometric phase contains an extra term in addition to the ordinary one proportional to the solid angle. For spin $1/2$ the extra term vanishes automatically. This may be the reason why it was not found previously. For higher spin, however, it depends on the initial condition and does not vanish in general. At this stage one may wonder when this extra term does not appear for an arbitrarily varying magnetic field. This is investigated in section 3. We prove that a sufficient condition is that the initial state is an eigenstate of $s \cdot e_0$ where s is the spin operator and e_0 is some unit vector. Though this conclusion is known in the literature [15, 16], our proof seems more straightforward and simpler.

In a recent paper, we studied a charged particle moving in a central potential plus a strong rotating magnetic field [17]. It can describe the valence electron of an alkaline atom or that of the hydrogen atom under the influence of the external magnetic field. The Schrödinger equation may be reduced to a Schrödinger-like one with a time-independent effective Hamiltonian by using an explicit time-dependent unitary transformation. Thus the evolution operator for the original Schrödinger equation was explicitly obtained, which involves no chronological product. Cyclic solutions are obtained if one takes the eigenstates of the effective Hamiltonian as initial states. These eigenstates and the nonadiabatic geometric phases of the corresponding cyclic solutions were all worked out explicitly. The nonadiabatic geometric phase turns out to be a linear combination of the two solid angles subtended by the traces of the orbit and spin angular momenta. We also studied the case without a central potential [17] and generalized it to the relativistic case [18].

Here we are interested in the more general cyclic solutions of the alkaline atomic electron in the strong rotating magnetic field. As pointed out in [17], these are available if the parameters of the system are appropriately chosen. However, the nonadiabatic geometric phases for such solutions were not calculated there. These are now calculated in section 4. The two solid angles subtended by the traces of the orbit and spin angular momenta are also calculated

explicitly. It turns out that the nonadiabatic geometric phase in this case also contains extra terms in addition to the linear combination of the two solid angles.

In section 5 we consider the alkaline atomic electron moving in an arbitrarily varying strong magnetic field. We prove that the nonadiabatic geometric phase for cyclic solutions with special initial conditions is a linear combination of the two solid angles. In other words, no extra term appears.

A brief summary is given in section 6. A formula used in the text is proved in the appendix.

2. Neutral particles in a rotating magnetic field

Before the calculations begin, let us remark here some differences between spin $1/2$ and higher ones. First, for any state of spin $1/2$, say, an initial state Ψ_0 , one can always find a unit vector e_0 such that $s \cdot e_0 \Psi_0 = \frac{1}{2} \Psi_0$. In fact, $e_0 = 2(\Psi_0, s\Psi_0)$ is the unit vector to be found. From this fact and the result of section 3, the previous conclusion for spin $1/2$ that the nonadiabatic geometric phase is always proportional to the solid angle follows immediately. For higher spin, on the other hand, the situation is rather different. For a given state Ψ_0 , in general one cannot find a unit vector e_0 such that $s \cdot e_0 \Psi_0 = m_s \Psi_0$ ($m_s = s, s-1, \dots, -s$). Let us give a simple example for spin $3/2$. We denote the eigenstate of s_z as $\chi_{m_s}^0$, with eigenvalue m_s . Now consider the state $\Psi_0 = a\chi_{3/2}^0 + b\chi_{-3/2}^0$, where $|a|^2 + |b|^2 = 1$ for normalization. The mean value of the spin in this state is $(\Psi_0, s\Psi_0) = (3|a|^2 - 3/2)e_z$, where e_z is the unit vector in the z -direction. By varying a , the absolute value of the above mean value may take any real number in the interval $[0, 3/2]$. Suppose that one could find a unit vector e_0 such that $s \cdot e_0 \Psi_0 = m_s \Psi_0$ ($m_s = \pm 3/2, \pm 1/2$), then the mean value of the spin would be $(\Psi_0, s\Psi_0) = m_s e_0$, and the absolute value is $|m_s|$, which is obviously in contradiction with that above. Second, even if the mean value of s in Ψ_0 is specified, say, $(\Psi_0, s\Psi_0) = m_s e_z$, one cannot assert that $s_z \Psi_0 = m_s \Psi_0$ (the inverse is of course true) unless $m_s = \pm s$ (this is automatically true for spin $1/2$). For example, for spin $3/2$, we have infinitely many states $\chi = \chi_{1/2}^0$ and $\chi' = e^{i\delta_1} \sqrt{2/3} \chi_{3/2}^0 + e^{i\delta_2} (1/\sqrt{3}) \chi_{-3/2}^0$ that lead to the mean value $\langle s \rangle = e_z/2$, where δ_1 and δ_2 are arbitrary real numbers.

Consider a uniform magnetic field $\mathbf{B}(t)$ that has a constant magnitude B and rotates around some fixed axis at a constant angle θ_B and with a constant frequency ω . The rotating axis is chosen as the z -axis of the coordinate system, so the magnetic field is

$$\mathbf{B}(t) = B\mathbf{n}(t) \quad \mathbf{n}(t) = (\sin \theta_B \cos \omega t, \sin \theta_B \sin \omega t, \cos \theta_B) \quad (1)$$

where B and ω are taken to be positive without loss of generality. Then consider a neutral particle with spin s ($s = 1/2, 1, 3/2, \dots$) and magnetic moment $\boldsymbol{\mu} = \mu\mathbf{s}/s$, where \mathbf{s} is the spin operator in units of \hbar , satisfying $[s_i, s_j] = i\epsilon_{ijk}s_k$. In the above magnetic field, it has the time-dependent Hamiltonian

$$H(t) = -\boldsymbol{\mu} \cdot \mathbf{B}(t) = -\epsilon(\mu)\hbar\omega_B \mathbf{s} \cdot \mathbf{n}(t) \quad (2)$$

where $\omega_B = |\mu|B/s\hbar$ is positive and $\epsilon(\mu)$ is the sign function. The motion is governed by the Schrödinger equation

$$i\hbar \partial_t \Psi = H(t)\Psi. \quad (3)$$

To solve this equation, we make a unitary transformation [2]

$$\Psi(t) = W(t)\psi(t) \quad W(t) = \exp(-i\omega t s_z) \quad (4)$$

then $\psi(t)$ satisfies a Schrödinger-like equation

$$i\hbar \partial_t \psi = H_{\text{eff}} \psi \quad (5)$$

where the effective Hamiltonian reads

$$H_{\text{eff}} = H(0) - \hbar\omega s_z = -\epsilon(\mu)\hbar\omega_B \mathbf{s} \cdot \mathbf{n}(0) - \hbar\omega s_z. \quad (6)$$

This effective Hamiltonian is time independent, so that the Schrödinger-like equation (5) is readily integrable. For convenience in the following, we define the new quantities

$$\omega_S = [\omega_B^2 + \omega^2 + 2\epsilon(\mu)\omega_B\omega \cos\theta_B]^{1/2} \quad (7a)$$

$$\sin\theta_S = \frac{\omega_B \sin\theta_B}{\omega_S} \quad \cos\theta_S = \frac{\omega_B \cos\theta_B + \epsilon(\mu)\omega}{\omega_S} \quad (7b)$$

$$\mathbf{n}_S = (\sin\theta_S, 0, \cos\theta_S). \quad (7c)$$

In terms of these new quantities, we have

$$H_{\text{eff}} = -\epsilon(\mu)\hbar\omega_S \mathbf{s} \cdot \mathbf{n}_S. \quad (8)$$

Therefore the Schrödinger-like equation (5) is solved as

$$\psi(t) = U_{\text{eff}}(t)\psi(0) \quad U_{\text{eff}}(t) = \exp[i\epsilon(\mu)\omega_S t \mathbf{s} \cdot \mathbf{n}_S]. \quad (9)$$

With the obvious relation $\Psi(0) = \psi(0)$, the Schrödinger equation (3) is solved as

$$\Psi(t) = U(t)\Psi(0) \quad (10a)$$

where

$$U(t) = W(t)U_{\text{eff}}(t) = \exp(-i\omega t s_z) \exp[i\epsilon(\mu)\omega_S t \mathbf{s} \cdot \mathbf{n}_S]. \quad (10b)$$

If one begins with an initial state $\Psi(t_0)$ at the time t_0 (but note that the time dependence of the magnetic field is still given by equation (1)), then the solution reads

$$\Psi(t) = W(t)U_{\text{eff}}(t - t_0)W^\dagger(t_0)\Psi(t_0). \quad (11)$$

Since the evolution operator involves no chronological product, it is convenient for practical calculations. In the following discussions we will take the initial time to be $t_0 = 0$ for convenience.

First of all let us calculate the mean value of \mathbf{s} in an arbitrary state. We define

$$\mathbf{v}(t) = (\Psi(t), \mathbf{s}\Psi(t)) \quad (12)$$

and denote $\mathbf{v}_0 = \mathbf{v}(0)$. Using equation (10) we have

$$\mathbf{v}(t) = (\Psi(0), U_{\text{eff}}^\dagger(t)W^\dagger(t)\mathbf{s}W(t)U_{\text{eff}}(t)\Psi(0)).$$

It is not difficult to show that

$$W^\dagger(t)\mathbf{s}W(t) = \exp(i\omega t s_z)\mathbf{s}\exp(-i\omega t s_z) = (s_x \cos \omega t - s_y \sin \omega t, s_x \sin \omega t + s_y \cos \omega t, s_z). \quad (13)$$

The following formula was proved in the appendix:

$$U_{\text{eff}}^\dagger(t)\mathbf{s}U_{\text{eff}}(t) = \exp[-i\epsilon(\mu)\omega_S t \mathbf{s} \cdot \mathbf{n}_S]\mathbf{s}\exp[i\epsilon(\mu)\omega_S t \mathbf{s} \cdot \mathbf{n}_S] \\ = [\mathbf{s} - (\mathbf{s} \cdot \mathbf{n}_S)\mathbf{n}_S] \cos[\epsilon(\mu)\omega_S t] - (\mathbf{n}_S \times \mathbf{s}) \sin[\epsilon(\mu)\omega_S t] + (\mathbf{s} \cdot \mathbf{n}_S)\mathbf{n}_S. \quad (14)$$

Using these two formulae we can calculate $\mathbf{v}(t)$ once \mathbf{v}_0 is given. Let us define

$$\mathbf{g}(t) = [\mathbf{v}_0 - (\mathbf{v}_0 \cdot \mathbf{n}_S)\mathbf{n}_S] \cos[\epsilon(\mu)\omega_S t] - (\mathbf{n}_S \times \mathbf{v}_0) \sin[\epsilon(\mu)\omega_S t] + (\mathbf{v}_0 \cdot \mathbf{n}_S)\mathbf{n}_S \quad (15)$$

which is an ordinary vector, not an operator. Then the result reads

$$\mathbf{v}(t) = (g_x(t) \cos \omega t - g_y(t) \sin \omega t, g_x(t) \sin \omega t + g_y(t) \cos \omega t, g_z(t)). \quad (16)$$

This is a rather complicated result, but the physical picture is clear. We observe that the three terms in $\mathbf{g}(t)$ are perpendicular to one another, and

$$|\mathbf{v}_0 - (\mathbf{v}_0 \cdot \mathbf{n}_S)\mathbf{n}_S| = |\mathbf{n}_S \times \mathbf{v}_0| = \sqrt{|\mathbf{v}_0|^2 - (\mathbf{v}_0 \cdot \mathbf{n}_S)^2} \equiv v_{0\perp} \quad (17)$$

so we define three unit vectors orthogonal to one another as

$$\mathbf{e}_x^S = [\mathbf{v}_0 - (\mathbf{v}_0 \cdot \mathbf{n}_S)\mathbf{n}_S]/v_{0\perp} \quad \mathbf{e}_y^S = \mathbf{n}_S \times \mathbf{v}_0/v_{0\perp} \quad \mathbf{e}_z^S = \mathbf{n}_S \quad (18)$$

which constitute a right-handed frame. In this frame $\mathbf{g}(t)$ takes the form

$$\mathbf{g}(t) = v_{0\perp} \cos[\epsilon(\mu)\omega_S t] \mathbf{e}_x^S - v_{0\perp} \sin[\epsilon(\mu)\omega_S t] \mathbf{e}_y^S + (\mathbf{v}_0 \cdot \mathbf{n}_S) \mathbf{e}_z^S. \quad (19)$$

This is nothing different from equation (15). However, in this form it helps us recognize the physical picture of the motion, and obviously yields $|\mathbf{v}(t)| = |\mathbf{g}(t)| = |\mathbf{v}_0|$ as expected. Now to get the vector $\mathbf{v}(t)$, one just rotates \mathbf{v}_0 around $\mathbf{e}_z^S = \mathbf{n}_S$ through an angle $-\epsilon(\mu)\omega_S t$ (a positive angle corresponds to anti-clockwise rotation) to get $\mathbf{g}(t)$, and then rotates $\mathbf{g}(t)$ around \mathbf{e}_z through an angle ωt . The resulting motion involves nutation as well as rotation, and the motion is not periodic in general. Note that a cyclic state leads to a periodic $\mathbf{v}(t)$, but the inverse is not necessarily true. Therefore, to obtain a cyclic solution, one should first find a periodic $\mathbf{v}(t)$. Two cases with periodic $\mathbf{v}(t)$ are available. First, if the parameters of the system are such that ω_S/ω is a rational number, then both ωt and $\omega_S t$ may simultaneously become integral multiples of 2π at some latter time T , and we have $\mathbf{v}(T) = \mathbf{v}_0$, independent of the initial condition. Second, if the initial condition is such that

$$\mathbf{v}_0 = m_s \mathbf{n}_S \quad m_s = s, s-1, \dots, -s \quad (20)$$

we have $\mathbf{g}(t) = m_s \mathbf{n}_S$ and

$$\mathbf{v}(t) = m_s (\sin \theta_S \cos \omega t, \sin \theta_S \sin \omega t, \cos \theta_S). \quad (21)$$

In this case $\mathbf{v}(t)$ only makes rotations. It is obviously periodic. In the following we see that both cases indeed correspond to cyclic solutions. It seems that no other cyclic solution can be found.

Cyclic solutions of the second kind (with special initial condition) have been previously discussed in detail [2]. For comparison we briefly review the result. First we give the eigenstates of $\mathbf{s} \cdot \mathbf{n}_S$. It is not difficult to show that

$$\mathbf{s} \cdot \mathbf{n}_S = \exp(-i\theta_S s_y) s_z \exp(i\theta_S s_y). \quad (22)$$

We denote the eigenstates of s_z with eigenvalues m_s ($m_s = s, s-1, \dots, -s$) as $\chi_{m_s}^0$. Then the eigenstates of $\mathbf{s} \cdot \mathbf{n}_S$ with eigenvalues m_s are obviously

$$\chi_{m_s} = \exp(-i\theta_S s_y) \chi_{m_s}^0 = \sum_{m'_s} D_{m'_s m_s}^s(0, \theta_S, 0) \chi_{m'_s}^0, \quad (23)$$

where the D 's are Wigner functions.

We take the initial state to be

$$\Psi_{m_s}(0) = \chi_{m_s}. \quad (24)$$

This leads to equations (20) and (21). The solution is indeed cyclic, and the nonadiabatic geometric phase in a period $\tau = 2\pi/\omega$ was found to be [2]

$$\gamma_{m_s} = -m_s \Omega_v \quad \text{mod } 2\pi \quad (25)$$

where $\Omega_S = 2\pi(1 - \cos \theta_S)$. We denote the solid angle subtended by the trace of $\mathbf{v}(t)$ by Ω_v . Because $\mathbf{v}(t)$ is given by equation (21) in the present case, we have

$$\Omega_v = \epsilon(m_s) \Omega_S \quad \text{mod } 4\pi$$

and consequently

$$\gamma_{m_s} = -|m_s|\Omega_v \pmod{2\pi}. \quad (26)$$

Therefore the geometric nature of the result is quite obvious. It should be remarked that the spin angular momentum precesses synchronously with the magnetic field, but at a different angle with the rotating axis.

Now we consider cyclic solutions of more general forms. If the parameters of the system are such that ω_S/ω is a rational number, then all solutions are cyclic, as shown below. We denote $\omega_S/\omega = K_S/K$, where K_S and K are natural numbers, prime to each other. Let $T = K\tau$, then $\omega T = 2\pi K$ and $\omega_S T = 2\pi K_S$. In this case $v(t)$ is periodic with period T . An arbitrary initial condition can be written as

$$\Psi(0) = \sum_{m_s} c_{m_s} \chi_{m_s} \quad (27)$$

where the coefficients c_{m_s} are arbitrary except for satisfying $\sum_{m_s} |c_{m_s}|^2 = 1$ such that $\Psi(0)$ is normalized. Note that $\Psi(0)$ can also be expanded in terms of the complete set $\{\chi_{m_s}^0\}$, it is easy to find that

$$\Psi(T) = \exp(i\delta)\Psi(0) \quad (28a)$$

where

$$\delta = s[\epsilon(\mu)2\pi K_S - 2\pi K] \pmod{2\pi}. \quad (28b)$$

Therefore, the state is indeed cyclic and δ is the total phase change which is independent of the initial condition. Using the relation

$$W^\dagger(t)H(t)W(t) = H(0) = H_{\text{eff}} + \hbar\omega s_z \quad (29)$$

noting that H_{eff} commutes with $U_{\text{eff}}(t)$ and s_z with $W(t)$, we have

$$\langle H(t) \rangle = (\Psi(0), H_{\text{eff}} \Psi(0)) + \hbar\omega v_z(t) = -\epsilon(\mu)\hbar\omega_S v_0 \cdot \mathbf{n}_S + \hbar\omega v_z(t).$$

Thus the dynamic phase is

$$\beta = -\hbar^{-1} \int_0^T \langle H(t) \rangle dt = \epsilon(\mu)2\pi K_S v_0 \cdot \mathbf{n}_S - 2\pi K \cos \theta_S v_0 \cdot \mathbf{n}_S. \quad (30)$$

This depends on the initial condition as expected. Finally we obtain the nonadiabatic geometric phase

$$\gamma = \delta - \beta = \epsilon(\mu)2\pi K_S (s - v_0 \cdot \mathbf{n}_S) - 2\pi K (s - \cos \theta_S v_0 \cdot \mathbf{n}_S) \pmod{2\pi}. \quad (31)$$

In the special case when $v_0 = m_s \mathbf{n}_S$; this is consistent with the previous result (note that $T = K\tau$).

The next task is to calculate geometrically the solid angle Ω_v subtended by the trace of $v(t)$, and compare it with γ . It is easy to show that

$$\Omega_v = \frac{1}{|v_0|} \int_0^T \frac{v_x(t)\dot{v}_y(t) - v_y(t)\dot{v}_x(t)}{|v_0| + v_z(t)} dt. \quad (32)$$

Because of the complicated results (15) and (16), it would be difficult to calculate this straightforwardly. Let us try to get around the difficulty. It is easy to show that

$$v_x(t)\dot{v}_y(t) - v_y(t)\dot{v}_x(t) = g_x(t)\dot{g}_y(t) - g_y(t)\dot{g}_x(t) + \omega [g_x^2(t) + g_y^2(t)]. \quad (33)$$

On account of the relations $|v(t)| = |g(t)| = |v_0|$ and $g_z(t) = v_z(t)$, we have $g_x^2(t) + g_y^2(t) = |v_0|^2 - v_z^2(t)$. Therefore

$$\Omega_v = \frac{1}{|g_0|} \int_0^T \frac{g_x(t)\dot{g}_y(t) - g_y(t)\dot{g}_x(t)}{|g_0| + g_z(t)} dt + \frac{1}{|v_0|} \int_0^T \omega [v_0 - v_z(t)] dt \quad (34)$$

where $\mathbf{g}_0 \equiv \mathbf{g}(0) = \mathbf{v}_0$. The second integral can be calculated easily, and the first is recognized as the solid angle subtended by the trace of $\mathbf{g}(t)$, which is very easy to calculate in the coordinate frame expanded by e_x^S , e_y^S and e_z^S (cf equation (19)). The final result is

$$\Omega_v = -\epsilon(\mu)2\pi K_S \left(1 - \frac{\mathbf{v}_0 \cdot \mathbf{n}_S}{|\mathbf{v}_0|}\right) + 2\pi K \left(1 - \cos\theta_S \frac{\mathbf{v}_0 \cdot \mathbf{n}_S}{|\mathbf{v}_0|}\right). \quad (35)$$

Here the first term is due to the rotation around $e_z^S = \mathbf{n}_S$, and the second is due to the further rotation around e_z . Compared with equation (31), we find the relation

$$\gamma = -|\mathbf{v}_0|\Omega_v + (s - |\mathbf{v}_0|)[\epsilon(\mu)2\pi K_S - 2\pi K] \pmod{2\pi}. \quad (36)$$

Therefore γ contains two terms. The first is the familiar one that is proportional to Ω_v . The second is an extra term. If $s = 1/2$, it is easy to show that $|\mathbf{v}_0| = 1/2$ for any initial state, then the extra term vanishes automatically, and the above relation reduces to $\gamma = -\frac{1}{2}\Omega_v$, which is known to be valid in an arbitrary magnetic field [13, 14]. For higher spin, $s - |\mathbf{v}_0|$ is in general not an integer, and the extra term cannot be dropped. For the special initial condition (24), the above relation reduces to the result (26). We will show in section 3 that equation (26) holds an arbitrarily varying magnetic field as long as the initial state is an eigenstate of $\mathbf{s} \cdot \mathbf{e}_0$ with eigenvalue m_s , where \mathbf{e}_0 is some unit vector. For the rotating magnetic field at hand, equation (26) holds as long as $|\mathbf{v}_0| = m_s$. This is a looser restriction on the initial condition. We do not know whether this is true in a more general magnetic field.

To conclude this section we remark that the relation (36) holds when $|\mathbf{v}_0| = 0$. This can be easily verified by comparing equation (36) with equation (31) in this special case. Moreover, from equation (30) we see that the dynamic phase vanishes. Therefore one may regard the total phase in this case as pure geometric, though Ω_v is not well defined.

3. Neutral particles in an arbitrarily varying magnetic field

As seen in section 2, the relation $\gamma \propto \Omega_v$ does not always hold for spin higher than $1/2$. Thus it may be of interest to ask when it would be valid in an arbitrarily varying magnetic field. In this section we will show that a sufficient condition is that the initial state is an eigenstate of $\mathbf{s} \cdot \mathbf{e}_0$, where \mathbf{e}_0 is some unit vector.

As discussed at the beginning of section 2, given an arbitrary state $\Psi(t)$ of spin $1/2$, one can always find a unit vector $\mathbf{e}(t)$ such that $\mathbf{s} \cdot \mathbf{e}(t)\Psi(t) = \frac{1}{2}\Psi(t)$. This holds at all times. For higher spin, however, no similar conclusion is available. Nevertheless, we will show that if an eigenvalue equation $\mathbf{s} \cdot \mathbf{e}_0\Psi(0) = m_s\Psi(0)$ holds for the initial state $\Psi(0)$, a similar one with some appropriate unit vector $\mathbf{e}(t)$ would hold at all later times. The latter equation is of crucial importance since it enables us to determine explicitly the state $\Psi(t)$ in terms of $\mathbf{e}(t)$ up to a phase factor. If $\mathbf{e}(t)$ returns to $\mathbf{e}(0)$ at some later time T , we obtain a cyclic solution.

We write down the Schrödinger equation in an arbitrarily varying magnetic field $\mathbf{B}(t) = B(t)\mathbf{n}(t)$,

$$i\hbar\partial_t\Psi = H(t)\Psi = -\hbar\omega_B(t)\mathbf{s} \cdot \mathbf{n}(t)\Psi. \quad (37)$$

There are two differences from the case in section 2. First, here $\omega_B(t) = \mu B(t)/s\hbar$ is time dependent, and its sign may change with time (so we do not use $|\mu|$ in defining $\omega_B(t)$). Second, the unit vector $\mathbf{n}(t)$ is not given by equation (1), but varies arbitrarily. We would assume that the magnetic field varies continuously.

We take the initial state $\Psi(0)$ of the system to be an eigenstate of $\mathbf{s} \cdot \mathbf{e}_0$ with eigenvalue m_s where \mathbf{e}_0 is some unit vector, that is

$$\mathbf{s} \cdot \mathbf{e}_0\Psi(0) = m_s\Psi(0) \quad m_s = s, s - 1, \dots, -s. \quad (38)$$

Let us define a vector $e(t)$ by the following differential equation and initial condition:

$$\dot{e}(t) = -\omega_B(t)\mathbf{n}(t) \times e(t) \quad e(0) = e_0. \quad (39)$$

Obviously, $|e(t)|$ is time independent, so $e(t)$ is a unit vector at any time. We are going to prove that

$$\mathbf{s} \cdot e(t)\Psi(t) = m_s \Psi(t) \quad (40)$$

holds at all later times. This can be easily done by induction.

By definition, equation (40) is valid at $t = 0$. We assume that it is valid at time t , what we need to do is to show that it is also true at time $t + \Delta t$ where Δt is an infinitesimal increment of time. In fact, using equations (37) and (39) we have

$$\Psi(t + \Delta t) = \Psi(t) + i\omega_B(t)\mathbf{s} \cdot \mathbf{n}(t)\Psi(t)\Delta t \quad (41a)$$

$$e(t + \Delta t) = e(t) - \omega_B(t)\mathbf{n}(t) \times e(t)\Delta t. \quad (41b)$$

After some simple algebra, the conclusion is achieved.

Because $e(t)$ is a unit vector, we can write

$$e(t) = (\sin \theta_e(t) \cos \phi_e(t), \sin \theta_e(t) \sin \phi_e(t), \cos \theta_e(t)). \quad (42)$$

It is not difficult to show that

$$\mathbf{s} \cdot e(t) = \exp[-i\phi_e(t)s_z] \exp[-i\theta_e(t)s_y]s_z \exp[i\theta_e(t)s_y] \exp[i\phi_e(t)s_z]. \quad (43)$$

Therefore the eigenstate of $\mathbf{s} \cdot e(t)$ with eigenvalue m_s is

$$\Psi(t) = \exp[i\alpha(t)] \exp[-i\phi_e(t)s_z] \exp[-i\theta_e(t)s_y] \chi_{m_s}^0 \quad (44)$$

where $\alpha(t)$ is a phase that cannot be determined by the eigenvalue equation. However, $\alpha(t)$ is not arbitrary. To satisfy the Schrödinger equation, it should be determined by the other variables $\theta_e(t)$ and $\phi_e(t)$. In fact, the above equation yields

$$(\Psi(t), \Psi(t + \Delta t)) = 1 + i\dot{\alpha}(t)\Delta t - im_s \cos[\theta_e(t)]\dot{\phi}_e(t)\Delta t. \quad (45)$$

On the other hand, from equation (41a) we have

$$(\Psi(t), \Psi(t + \Delta t)) = 1 + i\omega_B(t)\mathbf{v}(t) \cdot \mathbf{n}(t)\Delta t \quad (46)$$

where $\mathbf{v}(t)$ is defined by equation (12). Comparing the two results we obtain

$$\dot{\alpha}(t) = m_s \cos[\theta_e(t)]\dot{\phi}_e(t) + \omega_B(t)\mathbf{v}(t) \cdot \mathbf{n}(t). \quad (47)$$

The motion of $e(t)$ is determined by the magnetic field. If the magnetic field is such that $e(t)$ returns to its initial value at time T , that is

$$\theta_e(T) = \theta_e(0) \quad \phi_e(T) = \phi_e(0) + 2\pi K \quad (48)$$

where K is an integer, then we get a cyclic solution. In fact, it is easy to show that

$$\Psi(T) = \exp(i\delta)\Psi(0) \quad (49a)$$

where

$$\delta = \alpha(T) - \alpha(0) - 2\pi m_s K \quad \text{mod } 2\pi \quad (49b)$$

is the total phase change. Using equations (47) and (48), it is recast in the form

$$\delta = -m_s \int_0^T [1 - \cos \theta_e(t)]\dot{\phi}_e(t) dt + \int_0^T \omega_B(t)\mathbf{v}(t) \cdot \mathbf{n}(t) dt. \quad (50)$$

The second term is obviously the dynamic phase β . Therefore the nonadiabatic geometric phase is

$$\gamma = -m_s \Omega_e \quad \text{mod } 2\pi \quad (51)$$

where

$$\Omega_e = \int_0^T [1 - \cos \theta_e(t)] \dot{\phi}_e(t) dt \quad (52)$$

is the solid angle subtended by the trace of $e(t)$.

Finally note that $v(t)$ satisfies the same equation as $e(t)$, and $v_0 = m_s e_0$ which can be easily verified, we have $v(t) = m_s e(t)$. Consequently, equation (51) is recast in the form

$$\gamma = -|m_s| \Omega_v \pmod{2\pi}. \quad (53)$$

This is the final result of this section. Though γ and Ω_v cannot be calculated explicitly, the above relation holds regardless of the form of the magnetic field. For $m_s = \pm s$, this result was previously obtained in [15], and for general m_s in [16], both by different methods from ours, but our method seems more straightforward and simpler.

It should be noted that $v(t) = 0$ when $m_s = 0$. In this case Ω_v is not well defined. However, the final result (53) remains correct because it gives the same result $\gamma = 0$ as given by equation (51). This remark also applies to the result (26) in section 2 and similar ones in the following sections. To conclude this section, we remark that for any cyclic motion one can always choose the coordinate axes appropriately such that $\theta_e(t)$ does not take on the values 0 or π during the cycle under consideration. This avoids any non-continuous jump or ill definition of $\phi_e(t)$ and renders the above demonstration sound enough.

4. Alkaline atomic electron in a strong rotating magnetic field

In this section, we consider the valence electron of the alkaline atom or that of the hydrogen atom moving in a strong rotating magnetic field given by equation (1). This is described by the Schrödinger equation

$$i\hbar \partial_t \Psi = H(t) \Psi \quad (54a)$$

where

$$H(t) = H_0 + \mu_B B (\mathbf{l} + 2\mathbf{s}) \cdot \mathbf{n}(t) = H_0 + \hbar \omega_B (\mathbf{l} + 2\mathbf{s}) \cdot \mathbf{n}(t) \quad (54b)$$

in which

$$H_0 = \frac{\mathbf{p}^2}{2M} + V(r) \quad (54c)$$

is the Hamiltonian of the electron in the central potential of the nucleus (and the other electrons in the inner shells for alkaline atoms), M and μ_B are, respectively, the reduced mass and the Bohr magneton of the electron, $\omega_B = \mu_B B / \hbar > 0$, $\mathbf{l} = \mathbf{r} \times \mathbf{p} / \hbar$ is the orbit angular momentum operator in unit of \hbar and \mathbf{s} the spin as before (here $s = 1/2$). The applicability of this equation was discussed in [17].

The above Schrödinger equation can be solved in a way similar to that in section 2. This was discussed in detail in [17]. The solution is

$$\Psi(t) = U(t) \Psi(0) \quad U(t) = W(t) U_{\text{eff}}(t) \quad (55)$$

where

$$W(t) = \exp(-i\omega t j_z) \quad U_{\text{eff}}(t) = \exp(-iH_{\text{eff}} t / \hbar) \quad (56)$$

where j_z is the z -component of the total angular momentum (in units of \hbar) $\mathbf{j} = \mathbf{l} + \mathbf{s}$, and

$$H_{\text{eff}} = H_0 + \hbar \omega_L \mathbf{l} \cdot \mathbf{n}_L + \hbar \omega_S \mathbf{s} \cdot \mathbf{n}_S \quad (57)$$

is the effective Hamiltonian. The parameters in the effective Hamiltonian are defined as

$$\omega_L = (\omega_B^2 + \omega^2 - 2\omega_B\omega \cos \theta_B)^{1/2} \quad (58a)$$

$$\omega_S = (4\omega_B^2 + \omega^2 - 4\omega_B\omega \cos \theta_B)^{1/2} \quad (58b)$$

$$\mathbf{n}_L = (\sin \theta_L, 0, \cos \theta_L) \quad \mathbf{n}_S = (\sin \theta_S, 0, \cos \theta_S) \quad (59)$$

where

$$\sin \theta_L = \frac{\omega_B \sin \theta_B}{\omega_L} \quad \cos \theta_L = \frac{\omega_B \cos \theta_B - \omega}{\omega_L} \quad (60a)$$

$$\sin \theta_S = \frac{2\omega_B \sin \theta_B}{\omega_S} \quad \cos \theta_S = \frac{2\omega_B \cos \theta_B - \omega}{\omega_S}. \quad (60b)$$

Using the above solution we can calculate the mean values of \mathbf{l} and \mathbf{s} in an arbitrary state. We define

$$\mathbf{u}(t) = (\Psi(t), \mathbf{l}\Psi(t)) \quad \mathbf{v}(t) = (\Psi(t), \mathbf{s}\Psi(t)) \quad (61)$$

and denote $\mathbf{u}_0 = \mathbf{u}(0)$ and $\mathbf{v}_0 = \mathbf{v}(0)$. On account of the fact that \mathbf{l} , \mathbf{s} and H_0 commute with one another, the results can be obtained by calculations similar to those performed in section 2. We define

$$\mathbf{f}(t) = [\mathbf{u}_0 - (\mathbf{u}_0 \cdot \mathbf{n}_L)\mathbf{n}_L] \cos \omega_L t + (\mathbf{n}_L \times \mathbf{u}_0) \sin \omega_L t + (\mathbf{u}_0 \cdot \mathbf{n}_L)\mathbf{n}_L \quad (62a)$$

$$\mathbf{g}(t) = [\mathbf{v}_0 - (\mathbf{v}_0 \cdot \mathbf{n}_S)\mathbf{n}_S] \cos \omega_S t + (\mathbf{n}_S \times \mathbf{v}_0) \sin \omega_S t + (\mathbf{v}_0 \cdot \mathbf{n}_S)\mathbf{n}_S \quad (62b)$$

then the results read

$$\mathbf{u}(t) = (f_x(t) \cos \omega t - f_y(t) \sin \omega t, f_x(t) \sin \omega t + f_y(t) \cos \omega t, f_z(t)) \quad (63a)$$

$$\mathbf{v}(t) = (g_x(t) \cos \omega t - g_y(t) \sin \omega t, g_x(t) \sin \omega t + g_y(t) \cos \omega t, g_z(t)). \quad (63b)$$

Note that the second term in $\mathbf{g}(t)$ has a different sign from the one in section 2. The physical picture is rather similar to the previous one.

We denote the common eigenstates of $\{H_0, \mathbf{l}^2, l_z\}$ as ζ_{nlm}^0 , with eigenvalues $\{\epsilon_{nl}, l(l+1), m\}$, where ϵ_{nl} is the energy spectrum of the electron in the absence of the magnetic field. Then the common eigenstates of the operators $\{H_0, \mathbf{l}^2, \mathbf{l} \cdot \mathbf{n}_L, \mathbf{s} \cdot \mathbf{n}_S\}$ with eigenvalues $\{\epsilon_{nl}, l(l+1), m, m_s\}$ (now $m_s = \pm 1/2$) are

$$\varphi_{nlmm_s} = \zeta_{nlm} \chi_{m_s} \quad (64)$$

where χ_{m_s} is given by equation (23), and

$$\zeta_{nlm} = \exp(-i\theta_L l_y) \zeta_{nlm}^0 = \sum_{m'} D_{m'm}^l(0, \theta_L, 0) \zeta_{nlm'}^0. \quad (65)$$

The φ_{nlmm_s} are also eigenstates of H_{eff} with the eigenvalues

$$E_{nlmm_s} = \epsilon_{nl} + m\hbar\omega_L + m_s\hbar\omega_S. \quad (66)$$

However, these eigenvalues are not observable since H_{eff} is not a physical quantity. The above states are complete. At a given time, any state of the system can be expressed as a linear combination of them.

As shown in [17], a solution with the initial condition

$$\Psi_i(0) = \varphi_i = \varphi_{nlmm_s} \quad (67)$$

is a cyclic one. Here for convenience we use one subscript i to represent all the quantum numbers $nlmm_s$. The nonadiabatic geometric phase was shown to be

$$\gamma_i = -m\Omega_L - m_s\Omega_S \quad \text{mod } 2\pi. \quad (68)$$

Here, $\Omega_L = 2\pi(1 - \cos\theta_L)$ and $\Omega_S = 2\pi(1 - \cos\theta_S)$. We denote the solid angles subtended by the traces of the orbit and spin angular momenta by Ω_u and Ω_v , respectively. For the present case $\mathbf{u}(t)$ and $\mathbf{v}(t)$ have forms similar to equation (21), so we have

$$\gamma_i = -|m|\Omega_u - |m_s|\Omega_v \pmod{2\pi} \quad (69)$$

and the geometric nature of the result is quite obvious. For the alkaline atomic electron at hand, $|m_s|$ can be replaced by $1/2$ since $m_s = \pm 1/2$. However, the above result is valid for a charged particle of general spin s moving in the central potential plus the strong rotating magnetic field. In section 5 we show that this result holds in an arbitrarily varying strong magnetic field as long as the initial state is appropriately chosen. However, we will show in the following that for more general cyclic solutions the nonadiabatic geometric phase contains extra terms in addition to the linear combination of the two solid angles.

In the above we see that the initial condition $\Psi_i(0) = \varphi_i$ leads to a cyclic solution in the general case. If the parameters of the system satisfy some appropriate conditions, more cyclic solutions are available. In fact, if ω_B , ω and θ_B are such that both ω_L/ω and ω_S/ω are rational numbers, we see that any solution with the initial condition

$$\Psi(0) = \sum_{mm_s} c_{mm_s} \varphi_{nlmm_s} \quad (70)$$

is a cyclic one, where the coefficients c_{mm_s} are arbitrary except satisfying $\sum_{mm_s} |c_{mm_s}|^2 = 1$ such that the initial state is normalized. Suppose that $\omega_L/\omega = K_2/K_1$ and $\omega_S/\omega = K_4/K_3$, where all the K 's are natural numbers, with K_2 and K_1 prime to each other, and the same for K_4 and K_3 . We denote the least common multiple of K_1 and K_3 as K , and write

$$\omega_L/\omega = K_L/K \quad \omega_S/\omega = K_S/K \quad (71)$$

where $K_L = K K_2/K_1$ and $K_S = K K_4/K_3$ are both natural numbers. In this case, ωT , $\omega_L T$ and $\omega_S T$ are all integral multiples of 2π , where $T = K\tau$ is now the period of $\mathbf{u}(t)$ and $\mathbf{v}(t)$. It is not difficult to show that

$$\Psi(T) = \exp(i\delta)\Psi(0) \quad (72a)$$

where

$$\delta = -\epsilon_{nl}T/\hbar - l(2\pi K + 2\pi K_L) - s(2\pi K + 2\pi K_S) \pmod{2\pi}. \quad (72b)$$

Thus the solution is actually cyclic and δ is the total phase change which is independent of the initial condition. Currently, we would like to keep the general value s such that the result may be applied to a charged particle with more general spin. The dynamic phase can be calculated in a way similar to that in section 2, the result is

$$\beta = -\epsilon_{nl}T/\hbar - 2\pi K_L \mathbf{u}_0 \cdot \mathbf{n}_L - 2\pi K_S \mathbf{v}_0 \cdot \mathbf{n}_S - 2\pi K (\cos\theta_L \mathbf{u}_0 \cdot \mathbf{n}_L + \cos\theta_S \mathbf{v}_0 \cdot \mathbf{n}_S). \quad (73)$$

This depends on the initial condition as expected. The nonadiabatic geometric phase is

$$\begin{aligned} \gamma = \delta - \beta = & -2\pi K (l - \cos\theta_L \mathbf{u}_0 \cdot \mathbf{n}_L) - 2\pi K (s - \cos\theta_S \mathbf{v}_0 \cdot \mathbf{n}_S) \\ & - (l - \mathbf{u}_0 \cdot \mathbf{n}_L)2\pi K_L - (s - \mathbf{v}_0 \cdot \mathbf{n}_S)2\pi K_S \pmod{2\pi}. \end{aligned} \quad (74)$$

On the other hand, the solid angles subtended by the traces of the orbit and spin angular momenta are

$$\Omega_u = 2\pi K_L \left(1 - \frac{\mathbf{u}_0 \cdot \mathbf{n}_L}{|\mathbf{u}_0|}\right) + 2\pi K \left(1 - \cos\theta_L \frac{\mathbf{u}_0 \cdot \mathbf{n}_L}{|\mathbf{u}_0|}\right) \quad (75a)$$

$$\Omega_v = 2\pi K_S \left(1 - \frac{\mathbf{v}_0 \cdot \mathbf{n}_S}{|\mathbf{v}_0|}\right) + 2\pi K \left(1 - \cos\theta_S \frac{\mathbf{v}_0 \cdot \mathbf{n}_S}{|\mathbf{v}_0|}\right). \quad (75b)$$

The calculations that lead to these results are similar to those in section 2. From the above results and equation (74) it is easy to find that

$$\gamma = -|\mathbf{u}_0|\Omega_u - |\mathbf{v}_0|\Omega_v + (|\mathbf{u}_0| - l)(2\pi K + 2\pi K_L) + (|\mathbf{v}_0| - s)(2\pi K + 2\pi K_S) \pmod{2\pi}. \quad (76)$$

Therefore, in addition to a linear combination of Ω_u and Ω_v , we get two extra terms in γ . This result holds for a charged particle with spin s moving in the central potential plus the magnetic field. The first extra term can be dropped only when $|\mathbf{u}_0|$ is an integer and the second when $|\mathbf{v}_0| - s$ is an integer. For the alkaline atomic electron, $|\mathbf{v}_0| = s = 1/2$, so the second extra term vanishes. Because l is an integer, we have the final result for the alkaline atomic electron as

$$\gamma = -|\mathbf{u}_0|\Omega_u - \frac{1}{2}\Omega_v + |\mathbf{u}_0|(2\pi K + 2\pi K_L) \pmod{2\pi}. \quad (77)$$

For the special initial condition (67), the above results reduce to equation (69). In section 5 we will show that the result (69) holds in an arbitrarily varying magnetic field as long as the initial state is a common eigenstate of $\{H_0, \mathbf{l}^2, \mathbf{l} \cdot \mathbf{d}_0, \mathbf{s} \cdot \mathbf{e}_0\}$ with eigenvalues $\{\epsilon_{nl}, l(l+1), m, m_s\}$, where \mathbf{d}_0 and \mathbf{e}_0 are some unit vectors. For the rotating magnetic field at hand, it holds as long as $|\mathbf{u}_0| = |m|$ and $|\mathbf{v}_0| = |m_s|$ (the second is automatically satisfied for $s = 1/2$), which is a looser restriction on the initial state. We do not know whether this is true for a more general magnetic field. To conclude this section we point out that the condition (71) can be realized with $\omega_L/\omega = 1$ and $\omega_S/\omega = 2$, if one chooses $\omega_B/\omega = \sqrt{3}/2$ and $\cos\theta_B = \sqrt{3}/2\sqrt{2}$ [17].

5. Alkaline atomic electron in an arbitrarily varying strong magnetic field

Now we consider the alkaline atomic electron moving in an arbitrarily varying strong magnetic field. We write down the Schrödinger equation

$$i\hbar\partial_t\Psi = H(t)\Psi \quad (78a)$$

where

$$H(t) = H_0 + \hbar\omega_B(t)(\mathbf{l} + 2\mathbf{s}) \cdot \mathbf{n}(t). \quad (78b)$$

Compared with the case in section 4, there are two differences. First, $\omega_B(t) = \mu_B B(t)/\hbar$ is time dependent and may be either positive or negative. Second, $\mathbf{n}(t)$ is an arbitrarily varying unit vector. It is difficult to obtain any specific solution of the above equation. However, for solutions of special initial conditions, we can establish a relation between γ and the solid angles Ω_u and Ω_v .

Obviously, the operators in the set $\{H_0, \mathbf{l}^2, \mathbf{l} \cdot \mathbf{d}_0, \mathbf{s} \cdot \mathbf{e}_0\}$ commute with one another, where \mathbf{d}_0 and \mathbf{e}_0 are some unit vectors, thus they can have a complete set of common eigenstates. We take the initial state $\Psi(0)$ of the system to be such a common eigenstate, that is

$$H_0\Psi(0) = \epsilon_{nl}\Psi(0) \quad \mathbf{l}^2\Psi(0) = l(l+1)\Psi(0) \quad (79a)$$

$$\mathbf{l} \cdot \mathbf{d}_0\Psi(0) = m\Psi(0) \quad \mathbf{s} \cdot \mathbf{e}_0\Psi(0) = m_s\Psi(0). \quad (79b)$$

For the alkaline atomic electron, the last condition need not be assumed, since one can always find a unit vector \mathbf{e}_0 such that it holds with $m_s = 1/2$ or $-1/2$. However, we prefer to assume it so that the result may be valid for charged particles of more general spin. We define two vectors $\mathbf{d}(t)$ and $\mathbf{e}(t)$ by the following differential equations and initial conditions

$$\dot{\mathbf{d}}(t) = \omega_B(t)\mathbf{n}(t) \times \mathbf{d}(t) \quad \mathbf{d}(0) = \mathbf{d}_0 \quad (80a)$$

$$\dot{\mathbf{e}}(t) = 2\omega_B(t)\mathbf{n}(t) \times \mathbf{e}(t) \quad \mathbf{e}(0) = \mathbf{e}_0. \quad (80b)$$

Obviously, both $|\mathbf{d}(t)|$ and $|\mathbf{e}(t)|$ are time independent, so $\mathbf{d}(t)$ and $\mathbf{e}(t)$ are unit vectors at any time. One can prove that the following eigenvalue equations hold at all times:

$$H_0\Psi(t) = \epsilon_{nl}\Psi(t) \quad \mathbf{l}^2\Psi(t) = l(l+1)\Psi(t) \quad (81a)$$

$$\mathbf{l} \cdot \mathbf{d}(t)\Psi(t) = m\Psi(t) \quad \mathbf{s} \cdot \mathbf{e}(t)\Psi(t) = m_s\Psi(t). \quad (81b)$$

Because $\mathbf{d}(t)$ and $\mathbf{e}(t)$ are unit vectors, we can write

$$\mathbf{d}(t) = (\sin\theta_d(t)\cos\phi_d(t), \sin\theta_d(t)\sin\phi_d(t), \cos\theta_d(t)) \quad (82a)$$

$$\mathbf{e}(t) = (\sin\theta_e(t)\cos\phi_e(t), \sin\theta_e(t)\sin\phi_e(t), \cos\theta_e(t)). \quad (82b)$$

As in section 3, $\Psi(t)$ can be written as

$$\Psi(t) = \exp[i\alpha(t)] \exp[-i\phi_d(t)l_z] \exp[-i\theta_d(t)l_y] \zeta_{nlm}^0 \exp[-i\phi_e(t)s_z] \exp[-i\theta_e(t)s_y] \chi_{m_s}^0 \quad (83)$$

where $\alpha(t)$ is determined by the equation

$$\dot{\alpha}(t) = m \cos[\theta_d(t)] \dot{\phi}_d(t) + m_s \cos[\theta_e(t)] \dot{\phi}_e(t) - \hbar^{-1} \langle H(t) \rangle \quad (84)$$

where the expectation value $\langle H(t) \rangle$ is calculated in the state $\Psi(t)$.

The motions of $\mathbf{d}(t)$ and $\mathbf{e}(t)$ are determined by the magnetic field. If the latter is such that both $\mathbf{d}(t)$ and $\mathbf{e}(t)$ return to their initial values at time T , then we get a cyclic solution. The nonadiabatic geometric phase can be shown to be

$$\gamma = -m\Omega_d - m_s\Omega_e \quad \text{mod } 2\pi \quad (85)$$

where

$$\Omega_d = \int_0^T [1 - \cos\theta_d(t)] \dot{\phi}_d(t) dt \quad \Omega_e = \int_0^T [1 - \cos\theta_e(t)] \dot{\phi}_e(t) dt \quad (86)$$

are the solid angles subtended by the traces of $\mathbf{d}(t)$ and $\mathbf{e}(t)$.

By similar reasoning to that in section 3, we have $\mathbf{u}(t) = m\mathbf{d}(t)$ and $\mathbf{v}(t) = m_s\mathbf{e}(t)$ and equation (85) can be written as

$$\gamma = -|m|\Omega_u - |m_s|\Omega_v \quad \text{mod } 2\pi. \quad (87)$$

This is the final result of this section. It holds for a charged particle with spin s moving in the central potential plus the arbitrarily varying strong magnetic field. For the alkaline atomic electron, $|m_s|$ may be replaced by $1/2$.

6. Summary

In this paper we have studied the nonadiabatic geometric phases of neutral or charged particles moving in a time-dependent magnetic field. In section 2 we consider a neutral particle with general spin and with magnetic moment moving in a rotating magnetic field. The nonadiabatic geometric phase for special cyclic solutions is proportional to the solid angle subtended by the trace of the spin (cf equation (26)). This is well known. However, for more general cyclic solutions, we find that the nonadiabatic geometric phase contains an extra term. The main result of this section is equation (36). The extra term vanishes automatically for spin $1/2$, consistent with the known conclusion for spin $1/2$ that equation (26) is valid in an arbitrary magnetic field. For higher spin, however, the extra term depends on the initial condition. In section 3 we show that the result (26) is valid for special cyclic solutions of higher spin in an arbitrarily varying magnetic field. In section 4 we consider a charged particle moving in a central potential plus a strong rotating magnetic field. This may describe the valence electron in an alkaline atom or that in a hydrogen atom. For special cyclic solutions, the nonadiabatic

geometric phase is a linear combination of the two solid angles subtended by the traces of the orbit and spin angular momenta (cf equation (69)). This is also a previously known result. For the more general cyclic solutions, however, extra terms are also involved in the geometric phase. The main results of this section are equations (76) and (77). In section 5 we prove that the result (69) is valid for special cyclic solutions of charged particles moving in a central potential plus an arbitrarily varying strong magnetic field.

Acknowledgment

This work was supported by the National Natural Science Foundation of the People's Republic of China.

Appendix

Here we prove equation (14) in a very simple way.

We define

$$\mathbf{F}(\phi) = \exp(i\phi \mathbf{s} \cdot \mathbf{n}_S) \mathbf{s} \exp(-i\phi \mathbf{s} \cdot \mathbf{n}_S). \quad (\text{A1})$$

Differentiation of this equation with respect to ϕ yields

$$\mathbf{F}'(\phi) = \mathbf{n}_S \times \mathbf{F}(\phi) \quad (\text{A2})$$

and

$$\mathbf{F}''(\phi) = \mathbf{n}_S \times [\mathbf{n}_S \times \mathbf{F}(\phi)] = [\mathbf{F}(\phi) \cdot \mathbf{n}_S] \mathbf{n}_S - \mathbf{F}(\phi). \quad (\text{A3})$$

From equation (A2) we have $[\mathbf{F}(\phi) \cdot \mathbf{n}_S]' = \mathbf{F}'(\phi) \cdot \mathbf{n}_S = 0$, so that $\mathbf{F}(\phi) \cdot \mathbf{n}_S = \mathbf{F}(0) \cdot \mathbf{n}_S = \mathbf{s} \cdot \mathbf{n}_S$. Then equation (A3) becomes

$$\mathbf{F}''(\phi) + \mathbf{F}(\phi) = (\mathbf{s} \cdot \mathbf{n}_S) \mathbf{n}_S. \quad (\text{A4})$$

The solution of this equation is obviously

$$\mathbf{F}(\phi) = \mathbf{a} \cos \phi + \mathbf{b} \sin \phi + (\mathbf{s} \cdot \mathbf{n}_S) \mathbf{n}_S \quad (\text{A5})$$

where \mathbf{a} and \mathbf{b} are constant vectors. From equations (A1) and (A2) we have

$$\mathbf{F}(0) = \mathbf{s} \quad \mathbf{F}'(0) = \mathbf{n}_S \times \mathbf{s}. \quad (\text{A6})$$

This determines \mathbf{a} and \mathbf{b} , so we arrive at

$$\mathbf{F}(\phi) = [\mathbf{s} - (\mathbf{s} \cdot \mathbf{n}_S) \mathbf{n}_S] \cos \phi + (\mathbf{n}_S \times \mathbf{s}) \sin \phi + (\mathbf{s} \cdot \mathbf{n}_S) \mathbf{n}_S. \quad (\text{A7})$$

Equation (14) can be obtained by substituting $\phi = -\epsilon(\mu)\omega_S t$ into the above result.

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